WAVELET-BASED ANALYSIS OF GRAVITY GRADIOMETER DATA

Final Report

JPL Task 1031

Ulvi Yurtsever and Jeffrey Jewel, Exploration Systems Autonomy Section (367)

A. OBJECTIVES

To develop a wavelet-based algorithm for the fast-forward computation of the gravitational potential generated from arbitrarily complicated finite element models of the Earth's mass density. The primary motivation for the development of computationally efficient forward modeling is the future launch of sensitive gravity-gradiometer missions, which promise to return highly sensitive data with the potential of offering insight into the structure of the Earth's mass density. Traditional methods relate the observed gravitational potential to the estimated mass density through spherical harmonics. However, a decomposition in terms of these global-basis functions results in a decreased signal-to-noise for localized geophysical processes. What has not been fully exploited for both the forward and inverse solutions to the gravitational-inference problem is the use of spatially localized basis functions to represent the mass-density fluctuations. Such a wavelet-based expansion can provide a complementary language to the traditional use of spherical harmonics, and provide the computational ability to simulate gradiometer data as predicted from arbitrarily complicated finite element mass-density models of the Earth's interior.

B. PROGRESS AND RESULTS

We have developed a wavelet-based algorithm for the fast-forward computation of the gravitational potential generated from arbitrarily complicated finite element models of the Earth's mass density. The primary motivation for the development of computationally efficient forward modeling is the future launch of gravity-gradiometer missions which promise to return highly sensitive data with the potential of offering penetrating insight into the structure of the Earth's mass density. It is well known that inversion of the gravitational field for the mass density is an ill-posed problem, with a non-unique solution *in the absence of constraints*. We have instead focused on a fast-forward computational solution in the context of a Bayesian approach to the inverse problem, in which fluctuations in mass density about a detailed "virtual Earth" model can be well described as spatially localized basis functions (i.e. wavelets). Fast-forward-modeling capabilities will allow more realistic and detailed models of the Earth's mass density, and improve our ability to infer fluctuations in mass density about this model.

Gradiometer data can potentially provide a unique perspective on internal changes in the Earth and its response to the primary forcings of the Earth system as a whole – two of the fundamental problems of central importance to NASA's Earth Science Enterprise. The dynamic range of geophysical processes imprinting a detectable signature on the gravitational potential ranges from time scales on the order of milliseconds to millennia, and spatial scales from microscopic to continental. It has been demonstrated that the signature of great subduction earthquakes on the gravitational potential will be detectable with observations returned from GRACE [1]. In addition, the signature of mass motions occurring through mantle convection, magma movements, and ice-mass changes are other examples of processes potentially detectable with space-based gradiometry data.

Exploiting gradiometry data to probe the full spatio-temporal dynamic range of geophysical phenomena demands models of the density fluctuations well matched to the relevant physics. Traditional methods relate the observed gravitational potential to the estimated mass density through spherical harmonics. However a decomposition in terms of these global basis functions results in a decreased signal-to-noise for localized geophysical processes. While some mass-density fluctuations are in fact global in nature, and therefore efficiently matched with spherical harmonics, others are much more spatially local in extent. What has not been fully exploited for both the forward and inverse solutions to the gravitational inference problem is the use of spatially localized basis functions to represent the mass-density fluctuations. Such a wavelet-based expansion can provide a complementary language to the traditional use of spherical harmonics, and provide enhanced sensitivity for the detection of localized mass-density fluctuations. Furthermore, the simulation of data arising from a specific mass-density model can also benefit from a decomposition of the Earth's mass density as a linear combination of spatially localized functions.

In any basis chosen, inferring the mass-density fluctuations from gradiometry data is not a well-posed problem. However, there are physically reasonable constraints that can be imposed on solutions, allowing useful information to be extracted from the data. Specifically, the inverse problem may be regularized by the assumption that the fluctuations in mass density about initial models of the interior structure of the Earth are well understood statistically. This forms a Bayesian approach to the inverse problem, and a unique inversion of gradiometry data is possible with this type of statistical regularization – however the solution is a biased estimate of the mass density and, while a unique solution, not necessarily the true mass density. Of crucial importance then is precisely characterizing the "null space" of the proposed inverse algorithm.

B1. Overview

There are two main ideas, which will be explained in more detail in what follows, for the computational approach to fast computation of the gravitational potential from arbitrarily complicated finite element models of mass density. The first is the recognition that when fluctuations in mass density are expressed in a wavelet basis, the resulting gravitational field is exactly given by a linear combination of localized functions, such as a linear combination of Gaussians. This provides an efficient strategy when computing the gravitational potential from a finite element mass-density model. Given the wavelet decomposition of the mass density, we can compute the gravitational potential on a spherical surface with the use of a recently discovered fast convolution on the sphere [2]. The existence of the fast-convolution method on a sphere provides the ability to make simulations of GRACE data in the presence of complicated models, as well as quantify the ability to extract the signature of some particular geophysical process in the presence of other known signals.

An advantage of the use of the fast-convolution approach is the ability to compute the potential on many spherical surfaces. This will allow for the efficient computation of more realistic (ellipsoidal) orbits which are contained in a spherical shell. The method is also general enough to accommodate non-spherically symmetric wavelets (such as gradient wavelets), providing the ability to efficiently model a wide diversity of geophysical signals of interest.

B2. Wavelet Approach to the Analysis and Modeling of Gravity Data

Although the gravity-gradiometer inverse problem is non-unique, there is a diversity of relevant information that can be used to constrain solutions and extract interesting information from the data. We take the view that: 1) the solutions sought are *perturbations about some baseline model,* in which the gradient tensor field is predicted uniquely in the forward direction, 2) the perturbations in the mass density to be inferred from the data are modeled in terms of three-dimensional spatially localized functions in the Earth's interior. The expansion of the gravitational potential on the sphere with localized basis functions is directly related (as we will show) to the assumption of localized density fluctuations. This approach, other than the traditional spherical harmonic basis expansion of the mass-density fluctuations, can lead to a higher signal-to-noise for detection of *local fluctuations in the mass density* about the baseline model.

In more detail, we propose to model the mass-density interior to the Earth as the linear combination

$$\rho(r) = \rho_0(r) + \delta\rho(r)$$

where $\rho_0(r)$ is any assumed initial guess of the mass density, and $\delta\rho(r)$ are fluctuations about that guess. The solution of the gravitational potential from Laplace's equation can be written

$$\Phi_0(r) = \int dr'\, G(r,r')\,\rho_0(r')$$
$$\Phi(r) = \Phi_0(r) + \int dr'\, G(r,r')\,\delta\rho(r')$$

where $G(r,r')$ is the Green's function of Laplace's equation. The first equation above is solved uniquely in the forward direction from the assumed $\rho_0(r')$. The second equation is solved in the inverse direction for the mass-density fluctuations from the observed gravitational potential (or gradient tensor).

The fluctuations in mass density $\delta\rho(r')$ will be represented as a linear combination of localized functions. To provide some intuition about this approach, consider the particularly simple example given by a fictional planet which has uniform mass density other than a spherically symmetric hole at some depth. If the initial baseline model is taken to be a uniform-density sphere, the exact solution to the gravitational potential would simply be that given by a uniform-density sphere with a hole. The fluctuation in the gradient tensor would only involve the term from the hole, and the only two parameters of the problem would be the size and location of the hole. Therefore, the simpler a solution is, the better constrained it is by the data.

This example is obviously not realistic. However, it is relevant to consider localized fluctuations in mass density about some local mean density, particularly when we are interested in detecting changes in the mass density, as the result of geophysical activity. Given a partial understanding of the relevant geophysics in various regions in the earth, the type of fluctuations in mass density can be constrained. This illustrates the importance of the specific computational problem focused on in this proposal – that of forward modeling. If we are to limit the fluctuations in the mass density to a “sparse” collection of localized fluctuations, then much of the complexity of the mass density should be taken into account in a forward model, and these effects “subtracted” from the gradiometry data. Of course what we really seek is the computational means to incorporate arbitrary complexity in baseline models of the Earth’s interior, compare with observations, and then guess at improvements to the model. Therefore, fast and efficient forward solution for the gravitational potential for continuous, complicated mass densities is central to “inverting” gravity data.

In particular, the optimal language to use in modeling the density fluctuations comes directly from the physics of the problem. By a “local density fluctuation” we essentially mean a deviation of the mass density from a local average. Local averages can be defined according to

$$Local\,Average \propto \int dr\, e^{-(r-b)A(r-b)}\,\rho(r)$$

where the matrix $\mathbf{A}$ describes an ellipsoidal shape centered at location $\mathbf{b}$ characterized in the three principal orthogonal directions as $\mathbf{A} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. A fluctuation is then given roughly by the difference of the mass density at some point with the local average at the same point. More generally, we can characterize fluctuations of a certain scale according to the Laplacian of localized Gaussians

$$Fluctuation = \int dr\, \nabla^2 e^{-(r-b)A(r-b)}\rho(r)$$

For example, a hole at some location in the Earth's interior $\mathbf{b}$ is given by a circularly symmetric Gaussian with $\lambda_1 = \lambda_2 = \lambda_3$. Cylindrical fluctuations can be more accurately represented as a string of elongated Gaussian blobs.

Using the localized basis functions, we will assume the mass-density fluctuations can be written as a linear superposition

$$\delta\rho(\mathbf{r}) = \sum_i F(\mathbf{b_i})\left[\nabla^2 e^{-(\mathbf{r}-\mathbf{b_i})\mathbf{A}(\mathbf{r}-\mathbf{b_i})}\right]$$

where the mass-density fluctuation coefficients $F(\mathbf{b})$ are ultimately to be inferred from the gradiometer data. It can be shown, using the Laplacian of Gaussian blobs, that the resulting gravitational potential is given directly as a linear combination of Gaussians,

$$\Phi(\mathbf{r}) - \Phi_0(\mathbf{r}) = \sum_i F(\mathbf{b_i})\left[e^{-(\mathbf{r}-\mathbf{b_i})\mathbf{A}(\mathbf{r}-\mathbf{b_i})}\right]$$

Simply taking the gradient tensor of this equation gives

$$\nabla_\alpha \nabla_\beta \delta\Phi(\mathbf{r}) = \sum_i F(\mathbf{b_i})\left[\nabla_\alpha \nabla_\beta e^{-(\mathbf{r}-\mathbf{b_i})\mathbf{A}(\mathbf{r}-\mathbf{b_i})}\right]$$

The important property of the functions

$$\nabla_\alpha \nabla_\beta e^{-(\mathbf{r}-\mathbf{b_i})\mathbf{A}(\mathbf{r}-\mathbf{b_i})}$$

is that they provide the ability to characterize the fluctuations locally (to some chosen scale).

The expansion of the mass density of the Earth in a wavelet basis is completely general. Specifically, any mass-density model for the Earth can be convolved with the chosen wavelets at all positions and scales to compute the wavelet coefficients

$$F(\mathbf{b};\sigma) = \int dr\, \rho(\mathbf{r})\, \nabla^2 g(\mathbf{r} - \mathbf{b};\sigma)$$

which generalizes the above to allow for wavelets of any size. Although the wavelets discussed here are non-orthogonal (and in fact form an overcomplete basis), it can be shown that the mass density is uniquely reconstructed from these coefficients according to an inverse transform, which similarly gives the gravitational potential as an integration over the wavelet coefficients, convolved with Gaussian basis functions at various positions and scales.

B3. Forward Computation of the Gravitational Potential

The above discussion dealt with the computation of the gravitational potential everywhere as generated from the mass density expanded in a wavelet basis. In practice, we will only observe the gravitational potential along a surface, such as the flight path of an airplane or the orbit of a spacecraft. As the spacecraft moves along its orbit, it is able to map out the gravitational potential (or gradient tensor) over an entire path of the sphere. Computing the gravitational potential on a spherical surface enclosing the Earth has traditionally been approached by computing the spherical harmonic coefficients of the gravitational potential directly from the detailed mass-density model (the so-called Stokes Coefficients). The gravitational potential is then given directly by an inverse spherical harmonic transform. By contrast, what is proposed here is to exploit the wavelet decomposition of the mass density to formulate the computation of the gravitational potential on the sphere as a "convolution on the sphere," for which a fast algorithm has recently been discovered [2].

Briefly, the forward computation proceeds as follows: Restricting the gravitational potential to a sphere of radius R, we can write the gravitational potential in terms of the wavelet decomposition

$$\Phi(R\mathbf{n}) = \int d\Omega \int b^2 db\, F(b\mathbf{n}',\sigma)\, g(R\mathbf{n} - b\mathbf{n}';\sigma)$$

Interchanging the order of integration, and breaking up the integral above as a sum over discrete spherical shells, we can write

$$\Phi(R\mathbf{n}) = \sum_i b_i^2 \Delta b \int d\Omega\, F(b\mathbf{n}',\sigma)\, g(R\mathbf{n} - b\mathbf{n}';\sigma)$$

The main idea is to notice that each spherical shell for the Earth gives a contribution which can be computed as a convolution on the sphere,

$$(g \circ F)(\mathbf{n};b,\sigma) \equiv \int d\Omega\, F(b\mathbf{n}',\sigma)\, g(R\mathbf{n} - b\mathbf{n}';\sigma)$$

where now the coefficients within a specific spherical shell are thought of as a function on the sphere, and the localized basis function projected on the spherical surface the convolution kernel. The integral above is exactly of the form considered by [2]. For the case considered here, the convolution kernel is circularly symmetric, and localized about the direction defined by a unit vector from the center of the Earth in the direction of the centroid of the mass-wavelet mode.

The primary result in [2] is that such a convolution can be computed by first computing

$$C_{mm'm''} = \sum_l f_{lm}\, d^l_{mm'}(\pi/2)\, d^l_{m'm''}(\pi/2)\, g^*_{lm''}$$

where $d^l_{mm'}$ are known as the Wigner functions for which recursive function evaluations exist [4] (and for which these can be pre-computed and stored in memory), $g^*_{lm''}$ are the spherical harmonic coefficients of the convolution kernel, and f_{lm} are the spherical harmonic coefficients for the wavelet coefficients from a specific spherical shell projected on the sphere. We only need to compute the spherical harmonic transform for each spherical shell

$$f_{lm} = \int d\Omega\, F(b\mathbf{n};\sigma)\, Y_{lm}(\mathbf{n})$$

This can be computed quickly by interpolating the values of the wavelet coefficients in projection on the sphere, followed by fast spherical harmonic transform. These coefficients only need to be computed once for a given mass-density model in order to generate the gravitational potential on any spherical surface.

As shown in [2], the convolution integral is now given by inverse Fourier Transform

$$C(\theta,\varphi,\omega) = \sum_{m,m',m''=-L}^{L} C_{mm'm''} e^{im\varphi + im'\theta + im''\omega}$$

where the direction on the sphere has been parameterized by $\mathbf{n} = (\theta,\varphi)$ and ω parameterizes the orientation (which is irrelevant if a spherically symmetric wavelet is used for the mass density). Therefore the decomposition of the mass density of the Earth in a wavelet basis leads directly to the computation of the gravitational potential on spherical surfaces as convolution on the sphere, for which an effective mapping of FFT techniques from the plane to the sphere has been discovered [2].

C. SIGNIFICANCE OF RESULTS

The approach proposed here is fundamentally different from the traditional approach of computing the spherical harmonic coefficients directly from the mass-density model (without first taking the wavelet transform of the mass density). Briefly, the Stokes coefficients are computed according to [3]

$$C_{lm} + S_{lm} = \int r^2 dr d\Omega \, r^l Y_{lm}(\mathbf{n}) \rho(r\mathbf{n})$$

and the gravitational potential is then given by an inverse spherical harmonic transform. The primary drawback to this approach is the dependence of the Stokes coefficients on the order of the spherical harmonic mode itself. This results in a different weighting for the radial part of the integral for each multipole order. What makes this new approach fundamentally much faster is that the computation of the Stokes coefficients demand that for each multipole order, we integrate through the mass density with a radial moment depending on the order itself. This effectively requires that all the radial moments of the mass density are known. By contrast, as we will discuss shortly, the potential on the sphere can be computed from the wavelet coefficients with only one specific integration through the volume, at the various scales of the wavelet decomposition.

The wavelet formalism here also captures the dependence of the gravitational potential as a function of the altitude of the spherical surface – this dependence is entirely contained in the spherical harmonic coefficients of the projected Gaussian on the sphere. For a circularly symmetric kernel, there is no azimuthal dependence only requiring a Legendre polynomial decomposition. This suggests that the gravitational potential for spherical surfaces farther away can be generated recursively through a local averaging (low-pass filter) on spherical surfaces interior to a given radius. This can be made exact for a spherically symmetric wavelet by breaking up the sum over multipole order into sums over various spatial frequency bands. Then, for any given altitude, we can simply reweight each band in order to synthesize the correct convolution kernel. The weights can be computed as a function of the altitude of the spherical surface, and therefore provides an extremely fast way of computing the gravitational potential from an innermost spherical surface.

This capability is of primary importance when taking into account realistic orbits of a spacecraft. Almost all real orbits will lie on ellipsoidal surfaces (which also precess). However, we can quickly compute the predicted gravitational potential for many spherical surfaces that contain the orbit of the spacecraft – this will allow a more direct comparison of predictions from a specific mass-density model with observations, taking into account one of the largest complications in this type of analysis. This capability will also be useful in the analysis of data returned from spacecraft orbiting other planets.

Of particular importance is the ability to quickly compute the change in the gravitational field due to a localized fluctuation along a wavelet direction. At any point on the orbit, the response of the gradiometer is most strongly influenced by mass-density fluctuations in the region of the Earth's interior directly below the spacecraft. The ability to detect mass-density fluctuations can be enhanced if the response of the gravitational potential to fluctuations below the spacecraft and close by is characterized. If we compare the forward predictions with the observed potential and find a discrepancy, we can correct the mass-density model by varying the mass density along some wavelet direction (with some particular scale) at a position directly below the location in the orbit. Our computational framework allows this change to be integrated quickly into a new prediction for the gravitational potential as follows: The change to the coefficient is centered about one "point" in the Earth's interior, and therefore localized about a specific direction in projection. Therefore, the change in the spherical harmonic coefficients are immediately known in terms of the Wigner functions, and complex exponentials in terms of the Euler angles characterizing the direction on the sphere. These coefficients are then effectively filtered by multiplication by the coefficients for the kernel, and inverse Fourier transforming gives the resulting map of the potential on the spherical surface.

The traditional approach to relating the mass-density fluctuations to the observed fluctuations in the acceleration of gradient tensor is through a spherical harmonic decomposition of the gravitational potential. The coefficients of the spherical harmonic expansion of the field are related to the mass-density fluctuations. However, the drawback with the use of the spherical harmonics is that every coefficient receives a contribution from all mass-density fluctuations, since the spherical harmonic transform is given by integration over the entire sphere. The situation is equivalent to taking the Fourier transform of a point mass – all the Fourier coefficients will have the same amplitude, with a very specific phase given by the location of the point mass. With several point masses, the ability to disentangle the phases and recover the locations becomes more difficult.

D. FINANCIAL STATUS

The total funding for this task was $100,000 all of which has been expended.

E. PERSONNEL

No other personnel were involved.

F. REFERENCES

[1] R.S. Gross and B.F. Chao, "The Gravitational Signature of Earthquakes", Proceedings of GGG2000, October 2000.

[2] B.D. Wandelt and K.M. Gorski, "Fast Convolution on the Sphere," Phys. Rev. D, 63, December 2002.

[3] W.M. Kaula, Theory of Satellite Geodesy, Blaisdell, Waltham, MA, 1966.

[4] T. Risbo, Journal of Geodesy, 70, 383, 1996.

[5] U. Yurtsever, "Can you weigh the shape of a rock? Remarks on the gravitational inverse problem," JPL/QCT preprint, July 2000.